POISSON SIGMA MODELS AND DEFORMATION QUANTIZATION

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Abstract. This is a review aimed at a physics audience on the relation between Poisson sigma models on surfaces with boundary and deformation quantization. These models are topological open string theories. In the classical Hamiltonian approach, we describe the reduced phase space and its structures (symplectic groupoid), explaining in particular the classical origin of the non-commutativity of the string end-point coordinates. We also review the perturbative Lagrangian approach and its connection with Kontsevich's star product. Finally we comment on the relation between the two approaches.

1. Introduction

A Poisson manifold is a smooth manifold endowed with a Poisson bivector field, viz., a skew-symmetric contravariant tensor α of rank 2 satisfying the Jacobi identity

$$\alpha^{i[j}\partial_i\alpha^{kl]} = 0, \tag{1}$$

where $[\cdots]$ denotes a sum over cyclic permutations of the indices included in the square brackets. Using a Poisson bivector field one can define the Poisson bracket $\{\, f \,,\, g \,\} := \alpha^{ij}\partial_i f \partial_j g$ of any two smooth functions f and g. Two typical examples are $i)$ a symplectic manifold, and $ii)$ $M = \mathfrak{g}^*$ with $\mathfrak{g}$ a Lie algebra and $\alpha^{ij}(x) = f^{ij}_k x^k$, where f^{ij}_k are the structure constants in some basis.

The Poisson sigma model [8, 13] is a sigma model whose worldsheet is a connected, oriented, smooth 2-manifold (possibly with boundary) Σ and whose target is a (say, m-dimensional) Poisson manifold (M,α). The fields of the model are a map $X\colon \Sigma \to M$ together with a 1-form η, with $\eta(u)$, $u \in \Sigma$, taking values in the cotangent space of M at $X(u)$. We will use Greek indices for worldsheet coordinates and Latin indices for target coordinates. The action functional of the Poisson sigma model reads then

$$S(X,\eta) = \int_\Sigma \left[\eta_{\mu i}\partial_\nu X^i + \frac{1}{2}\alpha^{ij}(X)\eta_{\mu i}\eta_{\nu j}\right] \mathrm{d}u^\mu \mathrm{d}u^\nu. \tag{2}$$

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If Σ has a boundary, we choose the boundary conditions $v^\mu(u)\eta_{\mu i}(u) = 0$, $u \in \partial\Sigma$, for any vector v tangent to the boundary.

In Section 2 we review some results of [4] about the classical reduced phase space and its groupoid structure. In Section 3 we describe briefly the critical points of the action functional (2), its symmetries and its perturbative quantization and explain its connection with star products. For more details, we refer to [3]. Finally, we comment on the action of target diffeomorphisms and on the relation between Kontsevich's star product and Weinstein's program based on symplectic groupoids.

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2. Classical Hamiltonian approach

The action (2) describes the propagation of a topological open string. Let us choose locally a time coordinate $t = u^0$ and denote by $u = u^1$ the space coordinate. As we are describing an open string, u will belong to a closed interval I which we may as well identify with $[0, 1]$. Then the Lagrangian function corresponding to (2) reads

$$L(X, \zeta; \beta) = \int_I \left[-\zeta_i \dot{X}^i + \beta_i \left(X^{i\prime} + \alpha^{ij}(X)\zeta_j \right) \right] \mathrm{d}u, \tag{3}$$

with $\beta_i = \eta_{0i}$, $\zeta_i = \eta_{1i}$, $\dot{X} = \partial X/\partial t$ and $X' = \partial X/\partial u$. The first term in the Lagrangian tells us that ζ and X are canonically conjugated variables;[1] viz., we have the Poisson brackets

$$\left\{ \zeta_i(u) \, , \, X^j(v) \right\} = \delta_i^j \delta(u - v), \tag{4}$$

while all other Poisson brackets vanish. In the second term the new variable β appears. It has to be thought of as a Lagrange multiplier imposing the constraints

$$X^{i\prime} + \alpha^{ij}(X)\zeta_j = 0. \tag{5}$$

We will denote in the following by $\mathcal{C}$ the space of solutions to (5). Equivalently, we may define $\mathcal{C}$ as the common zero set of the functions

$$H_\beta = \int_I \beta_i \left(X^{i\prime} + \alpha^{ij}(X)\zeta_j \right) \mathrm{d}u, \tag{6}$$

for all β vanishing on the boundary of I (since $\beta_i = \eta_{0i}$ and η_{0i} is the component of η tangent to the boundary). It is now easy to check that the

[1] In order to make the discussion mathematically precise, one should actually introduce a structure of manifold on the space of fields. This requires specifying which kind of maps one takes into consideration. If one chooses the minimal conditions for (5) to make sense—i.e., ζ continuous and X 1-differentiable—, one can then give the space of fields the structure of an infinite-dimensional symplectic manifold locally modeled on a Banach space; in this case, most facts about finite-dimensional manifolds remain true.

H_βs are first-class constraints: viz., their Poisson brackets vanish on $\mathcal{C}$, i.e., upon using (5). In fact, let

$$\delta_\beta X^i(u) := \left\{ H_\beta\,,\, X^i(u) \right\}, \qquad \delta_\beta \zeta_i(u) := \left\{ H_\beta\,,\, \zeta_i(u) \right\}$$

denote the Hamiltonian vector field of H_β applied to the fields of the model: viz.,[2]

$$\delta_\beta X^i = -\alpha^{ij}(X)\beta_j, \tag{7a}$$

$$\delta_\beta \zeta_i = \beta_i' + \partial_i \alpha^{rs}(X)\zeta_r\beta_s. \tag{7b}$$

Then we have,

$$\delta_\beta \left(X^{i\prime} + \alpha^{ij}(X)\zeta_j \right) = (-\alpha^{ij}(X)\beta_j)' - \partial_l \alpha^{ij}(X)\alpha^{lr}(X)\beta_r\zeta_j + \\ + \alpha^{ij}(X)\beta_j' + \alpha^{ij}(X)\partial_j\alpha^{rs}(X)\zeta_r\beta_s,$$

which vanishes upon using (5) and (1). So we define the reduced phase space $\mathcal{G}$ as $\mathcal{C}$ modulo the symmetries (7).

2.1. **"Topological" nature of the model.** The absence of a Hamiltonian in (3) implies that the system has no dynamics, so it is invariant under diffeomorphisms in the time direction. As for the space direction, consider an infinitesimal diffeomorphism of the interval I, i.e., a vector field ν (which we may identify with a function on I) vanishing at the boundary. Its action on the field X is given by the Lie derivative

$$\mathrm{L}_\nu X^i = \nu X^{i\prime} \overset{\text{on } \mathcal{C}}{=} -\nu\alpha^{ij}(X)\zeta_j.$$

Thus, we can identify the action of ν with a symmetry (7a) generated by $\beta_i = \nu\zeta_i$. Also observe that $\mathrm{L}_\nu \zeta = (\nu\zeta)' = \beta'$ (ζ is a 1-form on I). Since $\zeta_r\beta_s$ is symmetric in r and s for our choice of β, it turns out that the action of ν on the pair of fields X and ζ corresponds to a symmetry transformation (7).[3] As a result, the model is invariant under worldsheet diffeomorphisms.

We also observe that the actual number of degrees of freedom is finite. Namely, one may show that the dimension of the tangent space of $\mathcal{G}$ is finite and equal to $2m$, with $m = \dim M$. In fact, let (X, ζ) be a representative of a class of solutions to (5) modulo (7). Consider infinitesimal perturbations ξ and ς of X and ζ respectively. The linearization of (5) then reads

$$\xi^{i\prime} + A^i_k \xi^k = -\alpha^{ij}(X)\varsigma_j,$$

[2] The expression on the right hand side of (7b) is clearly not covariant under target coordinate transformations. However, one can easily check that the error is proportional to the constraint (5). So on $\mathcal{C}$ the symmetries are well-defined.

[3] This construction actually works only if we consider smooth maps X and ζ which seems to be in contradiction with the assumptions of footnote 1. The point is that one can show that each class of solutions contains a smooth representative.

with $A^i_k(u) = \partial_k\alpha^{ij}(X(u))\zeta_j(u)$. This has a unique solution once $\xi^i(0)$ and ς are given:

$$\xi^i(u) = (V^{-1}(u))^i_k \left(\xi^k(0) - \int_0^u V^k_r(v)\alpha^{rs}(X(v))\varsigma_s(v)\,\mathrm{d}v\right), \tag{8}$$

where V is the path-ordered integral of A, i.e., the solution of $V^j_i(u)' = V^j_k(u)A^k_i(u)$, $V^j_i(0) = \delta^j_i$. The symmetry transformation (7a) does not change the boundary values of X since β vanishes on the boundary; so the m parameters $x^i := \xi^i(0)$ are well-defined. Now consider the linearization of the symmetry (7b) on ς: $\delta_\beta\varsigma_i = \beta'_i - \beta_s A^s_i$. It follows that $p_j := \int_0^1 \varsigma_i(u)(V^{-1}(u))^i_j\,\mathrm{d}u$ is another set of m invariants.

Thus, every infinitesimal perturbation (ξ,ς) determines uniquely $2m$ parameters (x,p). Conversely, given (x,p), we can first choose $\varsigma_i(u) = p_j(K^{-1})^j_i$, with $K^i_j = \int_0^1 (V^{-1}(u))^i_j\,\mathrm{d}u$, and then compute ξ using (8).

2.2. **"Noncommutativity" of the end-points.** Since β vanishes at the boundary, the boundary values of X are invariant under (7a). We set $x^i = X(0)$ and $y^i = X(1)$. We want to show that, using the Poisson bracket induced from (4) on $\mathcal{G}$, we get

$$\{\,x^i\,,\,x^j\,\} = \alpha^{ij}(x), \qquad \{\,y^i\,,\,y^j\,\} = -\alpha^{ij}(y), \qquad \{\,x^i\,,\,y^j\,\} = 0. \tag{9}$$

The main problem in this computation is that the functions $X^i(0)$ and $X^i(1)$ do not possess Hamiltonian vector fields. However, upon using (5), we can equivalently define the "regularized" versions

$$x^i = X^i(u) + \int_0^u \alpha^{ij}(X(v))\zeta_j(v)\,\mathrm{d}v,$$
$$y^i = X^i(u) - \int_u^1 \alpha^{ij}(X(v))\zeta_j(v)\,\mathrm{d}v,$$

for any $u \in (0,1)$. So we can compute

$$\{\,x^i\,,\,x^j\,\} = \left\{\,X^i(u) + \int_0^u \alpha^{ij}(X(v))\zeta_j(v)\,\mathrm{d}v\,,\,X^j(0)\,\right\} = \alpha^{ij}(X(0)),$$

and proceed similarly for the other identities in (9). (Observe that there is no need to regularize the second entry in the Poisson bracket as it is enough that the first entry admits a Hamiltonian vector field, and then one applies this field to the second entry).

2.3. **Composing solutions.** If we are given two solutions (X_1,ζ_1) and (X_2,ζ_2) to (5) such that $X_1(1) = X_2(0)$, we may compose them into a

new solution (X_3, ζ_3):

$$X_3(u) = \begin{cases} X_1(2u) & 0 \le u < \frac{1}{2}, \\ X_2(2u-1) & \frac{1}{2} \le u \le 1, \end{cases}$$

$$\zeta_3(u) = \begin{cases} 2\zeta_1(2u) & 0 \le u < \frac{1}{2}, \\ 2\zeta_2(2u-1) & \frac{1}{2} \le u \le 1. \end{cases}$$

To avoid a singularity at $u = 1/2$, we should assume that ζ vanishes at the boundary points;[4] one can indeed prove that each class $[(X, \zeta)] \in \mathcal{G}$ contains a representative with this property.

This way we have constructed a partially defined product $\bullet$ on $\mathcal{G}$; viz., we have a map $\bullet \colon \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$, where $\mathcal{G} \times_M \mathcal{G}$ is the space of pairs of classes of solutions with the end-point of the first coinciding with the starting point of the second. By the discussion of subsection 2.1, we obtain immediately that this product is associative; in fact, the two possible ways of combining three solutions are related by a reparametrization of the interval $[0, 1]$, which—as we have seen—can be obtained by a symmetry transformation. Moreover, for each point x of M, we have a very peculiar solution: viz., $X(u) \equiv x$ and $\zeta(u) \equiv 0$; this solution plays the role of a unit at x for the product $\bullet$. Finally, for every solution (X, ζ), we have an inverse solution $(\bar{X}, \bar{\zeta})$, with $\bar{X}(u) = X(1-u)$, $\bar{\zeta}(u) = -\zeta(1-u)$.

The existence of all the structures described in the previous paragraph is expressed by saying that $\mathcal{G}$ is a groupoid for M. Actually, in the lucky cases when $\mathcal{G}$ turns out to be a smooth symplectic manifold, one can further prove that $\mathcal{G}$ is a symplectic groupoid for M. This notion was introduced by Karasev [9], Weinstein [14] and Zakrzewski [16]. It means that some extra conditions are satisfied; the main ones are the identities displayed in (9) and the fact that $\{(a, b, c) \in \mathcal{G} \times_M \mathcal{G} \times \mathcal{G} : a \bullet b = c\}$ is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$, where $\bar{\mathcal{G}}$ as a manifold is the same as $\mathcal{G}$ but with opposite symplectic structure.

2.4. **The formal symplectic groupoid.** Locally one can use (7b) to set ζ constant. If one denotes by p this constant, then a point of the reduced phase space may be taken as a solution of

$$X^{i\prime}(u) + \alpha^{ij}(X(u))p_j = 0. \tag{10}$$

This set of equations defines globally a space $\mathcal{G}'$ that coincides only locally with our $\mathcal{G}$. This space $\mathcal{G}'$ was studied in [9, 10] and called the local symplectic groupoid: it always admits the structure of a smooth symplectic manifold (globally, despite of the name) but has an associative product that is defined only in a neighborhood of the constant solutions. This is the price to pay with respect to our $\mathcal{G}$, which has a globally defined groupoid structure (but may fail to be globally a manifold).

[4]This makes ζ_3 continuous and X_3 1-differentiable as required by footnote 1.

It is possible however to take an intermediate point of view and get a globally defined groupoid structure compatible with a globally defined smooth symplectic structure but in a formal sense. Namely, we may consider solutions to (5) modulo symmetries (7) as formal power series in α and its derivatives. We will denote this formal symplectic groupoid by $\mathcal{G}_{\text{form}}$. Let us describe briefly its construction. As a manifold $\mathcal{G}_{\text{form}}$ is just T^*M. In fact given a point $(x,p) \in T^*M$, we associate to it the pair (X,ζ), where $\zeta \equiv p$ and X is the formal solution of (10) with initial condition $X(0)=x$. Conversely, given a solution to (10), it is possible to construct a formal path of symmetries (7) that makes ζ constant. Then we set $x = X(0)$ and $p = \zeta$.[5] The symplectic structure can be easily discussed by changing coordinates $(x,p) \mapsto (y,p)$ with $y = \int_0^1 X(u)\ \mathrm{d}u = x + O(\alpha)$ where X is the solution corresponding to (x,p). These are Darboux coordinates; in fact, the symplectic structure on the space of maps corresponding to (4) is the differential of $\int_0^1 \zeta_i(u)\delta X^i(u)\ \mathrm{d}u$ (where δ denotes the differential on the space of fields). Restricting this one form to $\mathcal{G}_{\text{form}}$ one gets $p_i \mathrm{d}y^i$, whose differential is the symplectic form $\mathrm{d}p_i \mathrm{d}y^i$. Repeating verbatim the construction of subsection 2.3, we can define a product on $\mathcal{G}_{\text{form}}$ which will be denoted by the same symbol $\bullet$.

Let us now consider the case when M is a domain in $\mathbb{R}^m$. Then $\mathcal{G}_{\text{form}}$ is $M \times \mathbb{R}^m \ni (y,p)$, and we have two globally defined projections. We see then $\mathcal{G}_{\text{form}} \times \mathcal{G}_{\text{form}} \times \bar{\mathcal{G}}_{\text{form}}$ as a trivial bundle over $\mathbb{R}^m \times \mathbb{R}^m \times M$. The fact that the groupoid product defines a Lagrangian submanifold of $\mathcal{G}_{\text{form}} \times \mathcal{G}_{\text{form}} \times \bar{\mathcal{G}}_{\text{form}}$ can now be expressed in terms of a generating function; viz., there exists a function $F(p_1,p_2,y_3)$ on $\mathbb{R}^m \times \mathbb{R}^m \times M$ such that

$$\left(\frac{\partial F}{\partial p_1}, p_1\right) \bullet \left(\frac{\partial F}{\partial p_2}, p_2\right) = \left(y_3, \frac{\partial F}{\partial y_3}\right). \tag{11}$$

Following an idea of Weinstein [15],[6] we can define a "semiclassical" product on $C^\infty(M)[[\hbar]]$ by the formula

$$f \star_{\text{sc}} g(y) = \int \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}F(p_1,p_2,y)} f(y_1) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_1\cdot y_1} g(y_2) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_2\cdot y_2} \frac{\mathrm{d}^m y_1 \mathrm{d}^m p_1\, \mathrm{d}^m y_2 \mathrm{d}^m p_2}{(2\pi\hbar)^{2m}}, \tag{12}$$

[5]Notice however that the expression of p in terms of the original nonconstant ζ will be rather complicated, though in principle iteratively computable. It will involve a formal series of iterated integrals of ζ and of α and its derivatives, giving a sort of generalization of the analogue expansion of the holonomy in the case when M is the dual of a Lie algebra and ζ can be viewed as a connection.

[6]In terms of geometric quantization, we consider a trivial line bundle over $\mathcal{G}_{\text{form}} \times \mathcal{G}_{\text{form}} \times \bar{\mathcal{G}}_{\text{form}}$ with connection $A = (-y_1 \cdot \mathrm{d}p_1 - y_2 \cdot \mathrm{d}p_2 - p_3 \cdot \mathrm{d}y_3)/\hbar$. Then $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}F(p_1,p_2,y_3)}$ is covariantly constant when restricted to the Lagrangian submanifold determined by F. Moreover, it is covariantly constant along the polarization generated by $\partial/\partial y_1$, $\partial/\partial y_2$ and $\partial/\partial p_3$. On the other hand, if we choose $A = -y \cdot \mathrm{d}p/\hbar$ as a connection on $\mathcal{G}_{\text{form}}$, then $f(y)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\cdot y}$ is covariantly constant along the polarization $\partial/\partial p$ (which is transverse to the above $\partial/\partial y$).

with $p \cdot y := p_i y^i$. An explicit computation shows that $F(\hbar p_1, \hbar p_2, y_3) = \hbar y_3^i (p_1+p_2)_i + \frac{\hbar^2}{2}\alpha^{ij}(y_3)(p_1)_i(p_2)_j + O(\hbar^3)$, so that $f \star_{\mathrm{sc}} g = fg + \frac{\mathrm{i}\hbar}{2}\alpha^{ij}\partial_i f \partial_j g + O(\hbar^2)$. Moreover, one can check that $F(p_1, 0, y_3) = y_3 \cdot p_2$ and that $F(0, p_2, y_3) = y_3 \cdot p_2$ which implies $f \star_{\mathrm{sc}} 1 = 1 \star_{\mathrm{sc}} f = f$. Finally, the structure of F implies that $f \star_{\mathrm{sc}} g = \sum_{n=0}^{\infty} \hbar^n B_n^{\mathrm{sc}}(f,g)$ where the B_n^{sc}s are differential operators w.r.t. both entries.

Unfortunately, although $\star_{\mathrm{sc}}$ is defined using the associative groupoid product $\bullet$, nothing guarantees that it is associative. This happens in some lucky instances; e.g., in the case of the dual of a Lie algebra, the above construction yields $F(p_1, p_2, y_3) = y_3 \cdot \mathrm{CBH}(p_1, p_2)$, where CBH is the Campbell–Baker–Hausdorff formula. In general, we may expect that (12) is only an approximation to an associative product obtained by adding to F corrections in $\hbar$, as we will see in the next Section.

3. Perturbative Lagrangian approach

We now turn back to the action functional (2). Its critical points are solutions to the equations

$$\partial_\nu X^i + \alpha^{ij}(X)\eta_{\nu j} = 0,$$
$$\epsilon^{\mu\nu}\left(\partial_\mu \eta_{\nu i} + \frac{1}{2}\partial_i \alpha^{jk}(X)\eta_{\mu j}\eta_{\nu k}\right) = 0,$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric Levi-Civita tensor. There are particular critical points, which we will call trivial, consisting of a constant map $X \equiv x \in M$ together with $\eta \equiv 0$.

The symmetries of the action functional can be deduced from the symmetries (7) obtained in the Hamiltonian formalism. More precisely, we can introduce a ghost β (with $\beta(u)$ in the cotangent space of M at $X(u)$ and $\beta \equiv 0$ on the boundary) and the BRST operator δ acting as follows:

$$\delta X^i = -\alpha^{ij}(X)\beta_j, \tag{13a}$$
$$\delta \eta_{\mu i} = \partial_\mu \beta_i + \partial_i \alpha^{rs}(X)\eta_{\mu r}\beta_s. \tag{13b}$$

The BRST variations of β may be taken to be

$$\delta \beta_i = \frac{1}{2}\partial_i \alpha^{jk}(X)\beta_j \beta_k, \tag{13c}$$

so that δ squares to zero on shell. However, apart from some simple cases (e.g., when M is the dual of a Lie algebra), this BRST operator does not square to zero off shell. So one has to use the BV formalism instead. A simple way of applying it to this case is to use the AKSZ formalism [1] as explained in [12, 5]. We will not enter into details here. Only observe that the AKSZ method provides a solution to the classical master equation, but one has then to check that the quantum master equation is also satisfied. This is done in [3] by using a regularization that breaks the rotational symmetry of the disk. If however $\partial_i \alpha^{ij} = 0$ in given coordinates, this regularization is not

necessary, and one obtains a cyclic product: $\int (f\star g)h \, \mathrm{d}^m x = \int (g\star h) f \, \mathrm{d}^m x$, see [7].

The main observation now is that the boundary condition on β together with (13a) implies that for any function f on M we may define a BRST invariant observable $f(X(u))$ where u is any point on the boundary of Σ.[7] Let us consider the simplest topology, i.e., assume that Σ is a disk. After picking three points 0, 1 and ∞ on its boundary, we may define a product on $C^\infty(M)[[\hbar]]$ by

$$f \star g(x) = \langle \, f(X(1)) \, g(X(0)) \, \delta_x(X(\infty)) \, \rangle_0, \tag{14}$$

where δ_x is the delta function peaked at $x \in M$ and $\langle \ \ \rangle_0$ denotes the (perturbative) expectation value w.r.t. the action (2) around the trivial critical point.[8] It is not difficult to check that $f \star 1 = 1 \star f = f$, that $f\star g = fg + \frac{\mathrm{i}\hbar}{2}\alpha^{ij}\partial_i f \partial_j g + O(\hbar^2)$ and that in general $f\star g = \sum_{n=0}^\infty \hbar^n B_n(f,g)$ where the B_ns are differential operators w.r.t. both entries (bidifferential operators). Finally, the topological nature of the model allows us to move points on the boundary, and this leads to the associativity of $\star$. Thus, (14) defines a so-called star product [2].

In [3] this perturbative expansion has been studied mapping the disk to the upper half plane (the marked point ∞ being mapped to infinity) with the gauge fixing $g^{\mu\nu}\partial_\mu \eta_{\nu i} = 0$, where $g^{\mu\nu}$ is the Euclidean metric. It turns out to coincide with Kontsevich's formula [11] of which it provides a field-theoretical motivation. Let us briefly describe the Feynman diagrams of this expansion. A diagram of order n (i.e., producing a coefficient $\hbar^n$) contains n vertices in the upper half plane and two vertices at the points 0 and 1 on the boundary: we call the former internal vertices and the latter external vertices. From each internal vertex there depart two ordered, oriented edges. The head of an edge can then land on another internal vertex (internal edge) or an external vertex (external edge).[9] Each diagram then gives a bidifferential operator by putting an α on each internal vertex, putting the functions f and g on the external ones, and associating to each edge a partial derivative acting on the object it lands on and with its index contracted with the corresponding index of the α placed at its starting point. The bidifferential operator is then multiplied by a weight obtained by multiplying propagators corresponding to edges and integrating over the configuration space of all vertices. For more details, we refer to [11, 3].

Observe that the perturbative expansion requires to Taylor expand the Poisson bivector field α and the functions f and g, and this in turn requires

[7]More generally, one can consider observables of the same form but with u in the interior of Σ if f is a Casimir function, i.e., $\alpha^{ij}\partial_j f \equiv 0$.

[8]Taking into account all critical points may yield restrictions to the possible values $\hbar$ can assume.

[9]Tadpoles (i.e., edges that land on their own starting point) would yield singular contributions and have to be removed, as usual, by renormalization. Again this is not necessary if $\partial_i \alpha^{ij} = 0$ in given coordinates.

choosing local coordinates on M. Changing coordinates is then rather involved. In particular, if ξ is a vector field on M, its usual action (Lie derivative) on functions is not compatible with the star product (i.e., it is not a derivation). Its correct deformation (see [5]) is given by the formula

$$A_\xi(f)(x) = \langle\, f(X(1))\, \mathcal{O}_\xi\, \delta_x(X(\infty))\, \rangle_0 = \xi^i(x)\partial_i f(x) + O(\hbar),$$

where $\mathcal{O}_\xi$ is the BV observable whose classical part is $\int_\gamma \eta_{\mu i}(u)\xi^i(X(u))\,\mathrm{d}u$, where γ is a path that separates the disk into a component containing 1 and a component containing ∞. This deformed Lie derivative is a component of Kontsevich's quasiisomorphim [11], and using its properties one understands how to globalize the star product [11, 6].

We now want to discuss the relations between Kontsevich's star product (14) and its semiclassical (possibly nonassociative) counterpart (12). Assume M to be a domain in $\mathbb{R}^m$ and let $p \cdot y$ denote $p_i y^i$ for $y \in M$ and $p \in T^*_y M \simeq \mathbb{R}^m$. Define $\phi_p(y) = \exp(-\mathrm{i}p\cdot y/\hbar)$ and set

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{F}(p_1,p_2,y;\hbar)} := \phi_{p_1} \star \phi_{p_2}(y). \tag{15}$$

Then, by linearity, we can write

$$f \star g(y) = \int \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{F}(p_1,p_2,y;\hbar)} f(y_1)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_1\cdot y_1} g(y_2)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_2\cdot y_2}\, \frac{\mathrm{d}^m y_1 \mathrm{d}^m p_1\, \mathrm{d}^m y_2 \mathrm{d}^m p_2}{(2\pi\hbar)^{2m}}, \tag{16}$$

which is the correct (associative) generalization of (12). We observe that $\hat{F}$ has three interesting properties: *i*) it is a formal power series in $\hbar$ (i.e., no negative powers); *ii*) the function $F := \hat{F}|_{\hbar=0}$ is a generating function of a symplectic groupoid on T^*M with canonical symplectic structure—i.e., F satisfies (11)—; *iii*) the coefficients of the strictly positive powers in $\hbar$ correspond to Feynman diagrams containing internal loops (i.e., loops consisting only of internal edges).

The first and the third properties follow from the fact that a Feynman diagram contributing to $\phi_{p_1} \star \phi_{p_2}$ will carry a coefficient $\hbar^{n-e}$ if it has n internal vertices and e external edges. Observe moreover that $\hat{F}$ is defined in terms of only those Feynman diagrams that are still connected after deleting all external legs. This means that if the diagram has n vertices, it can have at most $n+1$ external legs, and this proves *i*). To prove *iii*) observe that strictly positive orders in $\hat{F}$ correspond to diagrams whose number of external legs is at most equal to the number of internal vertices; but this means that the number of internal edges is at least equal to the number of internal vertices, so that there must be an internal loop. Property *ii*), viz. equation (11), is instead obtained by computing in saddle-point approximation the products $(f \star g) \star h$ and $f \star (g \star h)$ which must be equal thanks to associativity.

4. Discussion

In this note we have reviewed some early results of ours, in particular [3] and [4], and described some related ideas. In Section 2, after a brief

description of the structures of the classical reduced phase space of the Poisson sigma model found in [4], we have discussed how to apply an old idea of Weinstein's [15] to define a star product, see (12). At this level however there is no clue that the product defined this way is associative. In Section 3, after recalling the results of [3] about the perturbative quantization of the Poisson sigma model and its relation with Kontsevich's formula [11], we have seen how to obtain in this context a correction to Weinstein's formula, see (16).

The natural question at this point is that if it had been possible to find directly the corrections to (12) to make it associative. A related idea stems from the observation that (12) looks like an expectation value of a lattice formulation of the Poisson sigma model with a particular gauge fixing. Namely, let us consider a lattice of three triangular plaquettes. To each plaquette we associate a point y of the domain M and to each edge not on the boundary we associate an element p of $\mathbb{R}^m$. We take then $-F$ as the plaquette action. Formula (12) is then obtained by setting the variable of one plaquette equal to the given y and by placing functions f and g in the two remaining plaquettes and setting the p-variable corresponding to the edge between the latter two plaquettes equal to zero ("gauge fixing"). Can this be reformulated by constructing a lattice BV action for the Poisson sigma model corresponding to the above description? It may then happen that this action does not satisfy the quantum master equation and that its iterative solution explains the higher order terms in $\hat{F}$. Another possibility is that, on such a small lattice, we cannot obtain associativity which might however be recovered by taking a bigger one. How big should it be? Does it have to be infinite as long as α is not linear, or can one expect to be able to adjust the lattice size according to degree of α in the given coordinates?

Finally, the definition of $\hat{F}$ in (15) has a nice interpretation in terms of quantum field theory; viz., we may absorb the terms $-p_1X(0)$ and $-p_2X(1)$ in the action (2). These singular terms can be regularized by removing a small neighborhood of 0 and 1 from the integration domain of the action and adjusting the boundary conditions of η consequently. The result is an expansion not around a trivial solution but around a solution that maps the negative real axis, the interval $(0, 1)$ and the interval $(1, \infty)$ to the three points y_1, y_2 and y, respectively, obeying the formula $(y_1, p_1) \bullet (y_2, p_2) = (y, p)$ (for a uniquely determined p).

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